SOLUTION OF PROBLEMS IN NUMBERS 11 AND 12, VOL. 1.

Solutions of problems in Nos. 11 and 12 have been received as follows:

From R. J. Adcock, 48; A. L. Baker, 44, 45 & 50; G. L. Dake, 41, 43 & 45; Theo. L. De Land, 41, 44, 45 & 50; A. B. Evans, 48 & 50; Henry Gunder, 42, 43, 44, 45 & 46; Phil. Hoglan, 41, 42 & 45 Max. Liporwitz, 41; Artemas Martin, 41, 42, 43, 44, 45 & 48; Walter Siverly, 43 & 50.

41.—"Given $x^2 + y = 7 \ldots$ (1), $\quad x + y^2 = 11 \ldots \ldots$ (2), to find the values of x and y."

SOLUTION BY MAX. LIPORWITZ, CRESCENT CITY, CAL.

Eliminating x^2 in (1) and y^2 in (2), we have

(3) $\qquad x^4 - 14x^2 + x + 38 = 0$, and

(4) $\qquad y^4 - 22y^2 + y + 114 = 0$.

On inspection, one of the roots of (3) is found to be 2; and, in like manner, one of (4) to be 3. Freeing (3) and (4) of these roots, the equations are

(5) $\qquad x^3 + 2x^2 - 10x - 19 = 0$, and

(6) $\qquad y^3 + 3y^2 - 13y - 38 = 0$. Now, making

(7) $\qquad x = z - \frac{2}{3}$, and

(8) $\qquad y = u - 1$, and substituting these values of x and of y in (5) and (6), we obtain

(9) $\qquad z^3 - \frac{34}{3}z - \frac{317}{27} = 0$, and

10) $\qquad u^3 - 16u - 23 = 0$.

Applying the method of circular functions to the solution of (9) and of (10), we find the roots of z and of u which latter substituted in (7) and (8), furnish the three additional roots of x and of y, so that

$$x = 2, \qquad 3.131313, \qquad - 3.283185, \qquad \text{or} - 1.848120,$$
$$\text{and} \quad y = 3, \qquad - 2.805118, \qquad - 3.779095, \qquad \text{or} \quad 3.584284.$$

SOLUTION BY L. REGAN, BOONSBORO, IOWA.

From (1) we get $x = \sqrt{(7 - y)}$; substitute this value for x in (2) and we have $y^2 + \sqrt{(7 - y)} = 11$, or $y^2 - 11 = - \sqrt{(7 - y)}$. Squaring both sides of this equation we get $y^4 - 22 y^2 + y + 114 = 0$, or, $y^4 - 13y^2 + 39y = 9y^2 + 38y - 114$; by adding $3y^3$ to each side and factoring we have $y(y^3 + 3y^2 - 13y - 38) = 3(y^3 + 3y^2 - 13y - 38)$; dividing by the common factor we get $y = 3$, and from (2) we get $x = 2$.

42. — "Find two integral numbers the difference of whose squares is a cube, and the difference of their cubes a square."

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

Let $ax^3 + b$ and $ax^3 — b$ denote the numbers.

The difference of their squares is $4abx^3$, which must be a cube, $= 8a^3x^3$, then $b = 2a^2$.

The difference of their cubes is $6a^2bx^6 + 2b^3$, which must be a square, or
$$3x^6 + 4a^2 = \square = (2x^3 — 2a)^2. \quad \text{Whence } x^3 = 8a.$$
Take $a = 1$, then $x = 2$, $b = 2$, and the numbers are 10 and 6.

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43.—"Let AB and AC be two lines intersecting each other at right angles in A, and D, any point given in position. Required the position of a line EF through D intersecting the lines AB and AC in E and F, so that $(DE)^2 + (DF)^2 = m^2$."

SOLUTION BY E. B. SEITZ, GREENVILLE, O.

Draw DH perpendicular to AB, and DK to AC. Put $DH = a$, $DK = b$, $\angle DEA = \theta$. Then $DE = a \operatorname{cosec} \theta$, and $DF = b \sec \theta$;

$\therefore a^2 \operatorname{cosec}^2\theta + b^2 \sec^2\theta = m^2$, or $a^2\left(\dfrac{1 + \tan^2\theta}{\tan^2\theta}\right) + b^2(1 + \tan^2\theta) = m^2$,

or $\qquad b^2 \tan^4\theta — (m^2 — a^2 — b^2)\tan^2\theta = — a^2$, whence

$$\tan^2\theta = \frac{m^2 — a^2 — b^2 \pm \sqrt{[(m^2 — a^2 — b^2) — 4a^2b^2]}}{2b^2}, \text{ or}$$

$$\tan \theta = \pm\frac{1}{2b}\left[\sqrt{(m + a — b)(m — a + b)} \pm \sqrt{(m + a + b)(m — a — b)} \right]$$

which determines the angle that the line EF makes with AB. There are, therefore, in general, four positions of the line.

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44.—"If a circle be divided into three equal parts by two parallel chords, find the perpendicular distance between the chords in terms of the radius."

SOLUTION BY PROF. J. SCHEFFER, KENYON COLLEGE, GAMBIER, O.

Let C be the center of the circle and AB a chord which cuts off its third part. Denoting the angle ACB by ψ, we obtain for the area of the segment, the expression $r^2\pi\psi \div 360 — \frac{1}{2}r^2 \sin \psi$. Hence we have the equation

$$\frac{r^2\pi}{390}\psi — \frac{r^2 \sin \psi}{2} = \frac{\pi r^2}{3}, \text{ or } \psi — \frac{180}{\pi} \sin \psi — 120 = 0,$$

an equation which pertains to that class of equations called transcendental equations. Such equations can only be solved by a process of approximation. Thus we find $\psi = 149° \ 16' \ 30''$. Hence the required distance equals $2r \cos \frac{1}{2}\psi = .52985r$.

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45. —"Required the area and sides of an obtuse angled triangle whose angles are to one another as 2, 3 and 7 and whose longest side equals 1. No logarithms to be used in the solution."

SOLUTION BY HENRY GUNDER, GREENVILLE, O.

We readily find the angles to be 30°, 45°, and 105°.

Demitting a perpendicular from the 105° $\angle$, and calling it x, then will the other sides and area be $2x$, $x\sqrt{2}$ and $\frac{1}{2}x$. From the two right triangles thus formed we get $x(\sqrt{3} + 1) = 1$, $\therefore x = \frac{1}{2}(\sqrt{3} - 1)$. Hence the sides are $\sqrt{3} - 1$, and $\frac{1}{2}(\sqrt{6} - \sqrt{2})$ and the area is $\frac{1}{4}(\sqrt{3} - 1)$.

46.—"The area of the piston of a steam engine is 1200 square inches, the length of stroke $8\frac{1}{2}$ ft., the pressure of steam upon the piston 32 lbs. per sq. inch and the number of strokes per min. 18. Required the number of cubic feet of water the engine will raise from a mine 60 fath. deep, the friction being estimated at 1 lb. per sq. inch plus the pressure of the atmosphere."

SOLUTION BY HENRY GUNDER.

The moving force $= 32 - 15.7225 = 16.5775$ lbs. per sq. inch.
Therefore $(1200 \times 16.2775 \times 18 \times 8\frac{1}{2}) \div (92.418 \times 360) = 133$ — cubic feet per minute.

This seems to me to be the common sense view of it, since there is nothing said about the expansion of the steam during the stroke of the piston, nor its cooling. [We inserted this question by special request, though it was obvious that the data are insufficient for a practical solution].

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47. — [No solution of this question has been received.—The inertia of a body in motion may be represented by the force that would be required to arrest the motion.

Let AB represent the piston, BD, $B'D'$ the connecting rod and CD, CD' the crank arm in two different positions.
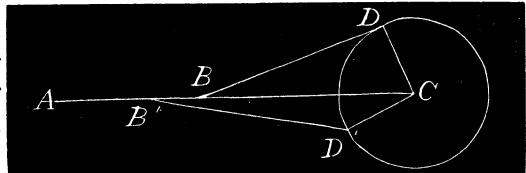
Then, if f represents the force acting upon the piston, θ the $\angle \ CBD$ and ϕ the $\angle \ BCD$, the relative moving force on a point in the axis of the

connecting rod will be represnted by the formula $f \cos \theta \sin \varphi$.

The inertia of a point in the axis of the connecting rod, and in the line of that axis, will therefore be a maximum when $\cos \theta \sin \psi$ is a maximum. The position of the connecting rod which corresponds to its maximum inertia in the line of its axis, depends, therefore, upon the relative length of the crank arm and the connecting rod.—ED.]

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48. — A cylindrical tower, radius r, is surrounded by a walk, width a. Two persons are on the walk; what is the probability that they can see each other?"

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

Let P be the position of one of the persons. Through P draw the diameter AP OB, and from P draw PG, PH tangent to the tower at C and D; then if the other person be on the surface $AEGCIDHFA$ they can see each other.
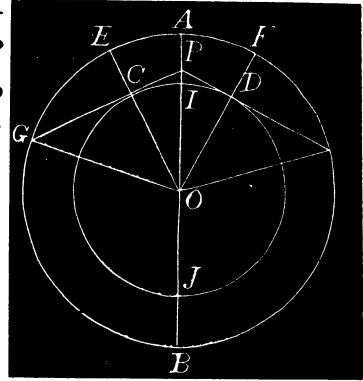
Let $OP = x$, then arc $EA = $ arc AF

$= (r + a) \cos^{-1}\left(\dfrac{r}{x}\right)$, arc $CI = $ arc ID

$= r \cos^{-1}\left(\dfrac{r}{x}\right)$, arc $GE = $ arc $FH = (r + a) \cos^{-1}\left(\dfrac{r}{r+a}\right)$,

$GC = DH = \sqrt{(2ar + a^2)}$; area $ECG = $ area $FDH = \frac{1}{2}(r + a)^2$

$\times \cos^{-1}\left(\dfrac{r}{r+a}\right) - \frac{1}{2}r\sqrt{(2ar + a^2)}$, area $EACI = $ area $AFID = \frac{1}{2}(r+a)^2$

$\times \cos^{-1}\left(\dfrac{r}{x}\right) - \frac{1}{2}r^2 \cos^{-1}\left(\dfrac{r}{x}\right) = \frac{1}{2}(2ar + a^2)\cos^{-1}\left(\dfrac{r}{x}\right)$.

$\therefore p =$

$$\frac{\displaystyle\int_{r}^{r+a}\left[(r+a)^2\cos^{-1}\left(\frac{r}{r+a}\right) + (2ar+a^2)\cos^{-1}\left(\frac{r}{x}\right) - r\sqrt{(2ar + a^2}\right]2\pi x\,dx}{\pi(2ar + a^2)\displaystyle\int_{r}^{r+a} 2\pi x\,dx}$$

$$= \frac{2(a + r)^2\cos^{-1}[r \div (a + r)]}{\pi(2ar + a^2)} - \frac{2r}{\pi\sqrt{(2ar + a^2)}}.$$

49.—"How far will a man travel in unwinding an inch rope from a frustrum of a cone whose upper diameter is 2 ft., lower 15 ft. and height 35 ft. the rope to be closely wound around the frustrum from top to bottom?"

Solution by Prof. *J. Scheffer,* Gambier, Ohio.

The area of the surface of the frustrum is $\pi(R + r)\sqrt{[h^2 + (R + r)^2]}$, R denoting the radius of the lower base, r that of the upper and h the axis, of the cone. The question arises now; what must be the length of a rectangular strip whose breadth is one inch, in order that its area may be equal to the above stated surface? Thus we find for the required distance:

$12\pi(R + r) \sqrt{[h^2 + (R + r)^2]} = 7616.6\pi = 23928.3$ ft., or nearly $1\frac{1}{2}$ miles.

This solution is not mathematically accurate. The line which the rope forms is not a curve (spiral or screw line) of double curvature, in fact it is no mathematical curve at all whose length we could find by means of the formulas of Calculus. The distance, however, which has been found in the above simple and elementary way, will not differ by a great deal from the *actual* one.

[It will be seen that, in the above solution, Prof. Scheffer has estimated approximately, the length of the *rope*, (or, rather, double the length, as, in substituting for R and r he has probably used the *diameters* of the bases instead of their radii,) but he has not attempted the more difficult part of the question, viz; to find the length of the involute curve traced on the ground in unwinding the rope.

Let r represent the distance from the apex of the cone to any point P, on its surface where the distance between the centers of two consecutive coils is x; and let θ represent the angle traversed by the radius vector r in describing that part of the curve which lies above the point P, then is $r = \int dxb\theta$ the polar equation of the curve. If x were constant this equation would represent the Spiral of Archimedes. For the given frustrum x is nearly constant and nearly equal to one inch, but not quite.

Determine x in functions of θ; then is $r = \int d.\psi(\theta)$. . . . . . (1)
Fro n (1) determine the length of the curve between the limits $r = h$ and $r = k$. (h representing the slant height of the complete cone and k the distance, on the slant side, from the apex of the cone to any point P, to which the rope may have been unwound). Let L represent the length between the above named limits and let p represent the perpendicular height of the point P; then will $\sqrt{(L^2 + p^2)} =$ the radius of curvature of the involute curve, from which its length may be found.

As a first step in the actual solution of prob. 49 we propose the question: To find the equation between x and θ as involved in (1) above.—Ed.]

50.—"Assuming the earth's orbit to be a circle, if a comet move in a parabola around the sun and in the plane of the earth's orbit, show that the comet cannot remain within the earth's orbit longer than 78 days."

Solution by Prof. *A. B. Evans,* Lockport, N. Y.

Let MAN be the orbit of a comet whose time within the earth's orbit $BEDF$ is a maximum. Let A be the comet's position when nearest the sun S, and C any other position of the comet in its orbit.

Put $SE = r$, $SA = a$, $SC = \rho$, $\angle ASC = \theta$: then the equation of the parabola is

$$\rho = \frac{2a}{1 + \cos \theta} = \frac{a}{\cos^2 \frac{1}{2}\theta}, \text{ and the area}$$

ASC is $\frac{1}{2}\int \rho^2 d\theta = \frac{1}{2}a^2 \int \dfrac{d\theta}{\cos^4\frac{1}{2}\theta}$

$$= \tfrac{1}{2}a^2 \int (1 + \tan^2 \tfrac{1}{2}\theta)\sec^2\tfrac{1}{2}\theta d\theta$$

$$= a^2\tan \tfrac{1}{2}\theta + \tfrac{1}{3}a^2\tan^3\tfrac{1}{2}\theta. \quad . \quad . \quad . \ (1)$$

Since $\cos^2\frac{1}{2}\theta = a \div \rho$, the area of ASD is found by putting $\cos \frac{1}{2}\theta = \sqrt{(a \div r)}$, or $\tan \frac{1}{2}\theta = \sqrt{[(r - a) \div a]}$, in (1). The area of ASD is therefore equal to $a^2\sqrt{[(r - a) \div a]} + \frac{1}{3}a^2\sqrt{[(r - a)^3 \div a^3]}$
$= \frac{1}{3}\sqrt{a}\times(r + 2a)\sqrt{(r - a)}$; and the area of $BSA + ASD = 2ASD$
$$= \tfrac{2}{3}\sqrt{a}\times(r - 2a)\sqrt{(r - a)}. \ldots (2)$$

Now the areas described about a common center of force by two bodies moving in different orbits being in the subduplicate ratio of the parameters of those orbits, and the parameters of the orbits in this case being $2r$ and $4a$, we have $\sqrt{(2r)} : \sqrt{(4a)} :: \pi r^2 : \pi\sqrt{r^3} \times \sqrt{(2a)} = $ the area described by the comet in one year. As the comet describes equal areas in equal times, we have, denoting the number of days in the year by T,

$\pi \sqrt{r^3} \sqrt{(2a)} : \frac{2}{3}\sqrt{a}.(r + 2a)\sqrt{(r - a)} :: T : T'$ the number of days the comet is within the earth's orbit

$$\therefore \ T' = \frac{(r + 2a)\sqrt{(2r - 2a)}}{3\pi\sqrt{r^3}} \ T \ . \ . \ . \ . \ (3)$$

The only variable in the value of T' being a, T' will be a maximum when $(r + 2a)\sqrt{2r - 2a}$ is a maximum. Differentiating this expression with respect to a and placing the differential coefficient equal to zero, we find $2\sqrt{(2r - 2a)} - (r + 2a) \div \sqrt{(2r - 2a)} = 0$; whence $r = 2a$. This value of r substituted in (3) gives

$$T' = \frac{2}{3\pi}.T = \frac{2}{3}\left(\frac{365.2563612}{3.14159}\right) = 77.5 \text{ days} \ \therefore \ T' < 78 \text{ days}.$$